THE SECOND LINK

The Random Universe

ON SEPTEMBER 18, 1783, IN ST. PETERSBURG Leonard Euler started the day as usual. He gave a mathematics lesson to one of his grandchildren and took up some calculations on the flight of balloons. Just three months earlier, south of Lyon, the Montgolfier brothers had launched an enormous balloon that rose 6,500 feet into the air and landed safely about a mile away. Euler was working out the mechanics of the balloon's motion as the Montgollier brothers were preparing to launch a sheep into the air in front of King Louis XVI in Paris, a flight that took place the next day, on September 19. Euler never heard about the event, however. After lunch, working with his assistants, he made some calculations on the orbit of the recently discovered planet Uranus. The equations introduced by him, capturing the planet's peculiar orbit, would lead decades later to the discovery of the planet Pluto. Euler did not live to witness that discovery either. About five o'clock in the afternoon, he suffered a brain hemorrhage and uttered. "1 am dying." before losing consciousness. He died that evening, ending the most prolific career in mathematics of all time.

Euler, a Swiss born mathematician who spent his career in Berlin and St. Petersburg, had an extraordinary" influence on all areas of mathematics, physics- and engineering. Not only was the importance of his discoveries unparalleled, their sheer quantity is also overwhelming. *Opera Omnia,* the still incomplete record of Euler's collected works, currently runs to over seventy-three volumes, six hundred pages each. The last seventeen year- of Baler's life, between his return to St. Petersburg in 1766 and his death at the age of 76, were rather tumultuous. Yet despite many personal tragedies, about half of his works were written during these years. These include a 775-page treatise on the motion of the moon, an influential algebra textbook, and a three-volume discussion of integral calculus, completed while he continued to publish an average of one mathematics paper per week in the journal of the St. Petersburg Academy. The amazing thing is that he barely wrote or read a single line during this time. Having partially lost his sight soon after returning to St. Petersburg in 1766. Euler was left completely blind after a failed cataract operation in 1771- The thousand? of pages of theorems were all dictated from memory.

Three decades earlier, his sight intact, Euler had written a short paper addressing an amusing problem that originated in Konigsberg, a town nor too far from Euler's home in St. Petersburg. Konigsberg, a flowering city in eastern Prussia, did not suspect in the early eighteenth century the sad and war-torn fate that awaited it as host for one of the fiercest battles of the Second World War. Contemporary etchings show a thriving city on the banks of the Pregel, where a busy fleet of ships and their trade offered a comfortable life to the local merchants and their families. The healthy economy allowed city officials to build not fewer than seven bridges across the river. Most of these connected the elegant island Kneiphof, which was caught between the two branches of the Pregel, with other parts of the city. Two additional bridges crossed the two branches of the river (Figure 2.1). The people of Konigsberg, enjoying a time of peace and prosperity, amused themselves with mind puzzles, one of which was: "Can one walk across the seven bridges and never cross the same one twice?" No one was to find such a path until a new bridge was built in 1S75.

Almost 150 years before the new bridge, in 1736, Euler offered a rigorous mathematical proof stating that with the seven bridges such a path does not exist. He not only solved the Konigsberg problem but in his brief paper inadvertently started an immense branch of mathematics known as graph theory. Today graph theory is the basis for our thinking about networks. During the centuries after Euler it grew into a mature field, to which most great mathematicians contributed. To open the door on the field of networks, let us briefly revisit the reasoning process that led Euler to the introduction of the first graph.

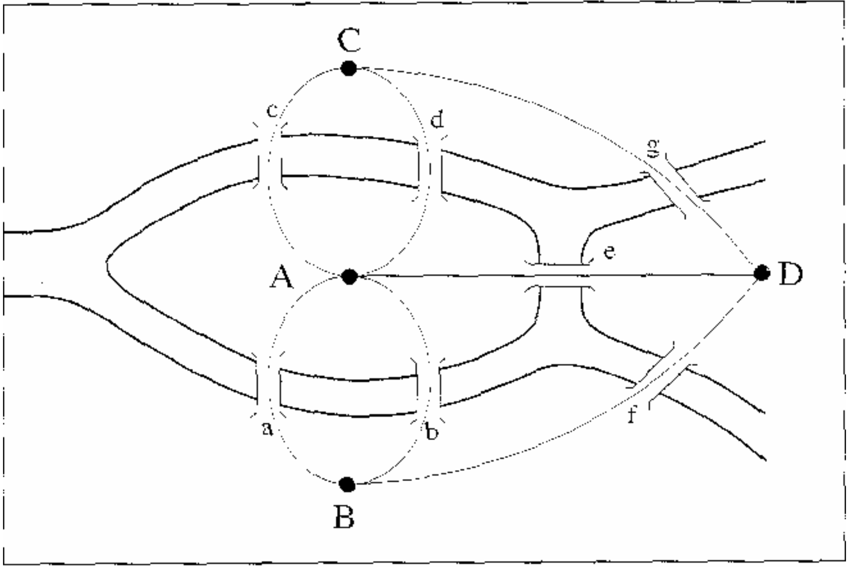

Figure 2.1 Konigsberg Bridges. *The layout of Konigiberg before 1875, with Kntiphof island(A) and the land area.* D *caught between the two branches of the Psegel River. Solving the Konigsberg problem meant finding a route around the city that would require a person to cross each bridge only once. In 1736, Leonard Euler gave birth to graph theory by replacing each of the four land areas with nodes (A to* D) *and each bridge with a link (*a *to* g), *obtaining a graph with four nodes and seven links. He then proved that on the Konigsberg graph, a route crossing each link. only once does not exist.*

1.

Euler's proof is simple and elegant, easily understood even by those not trained in mathematics. Nevertheless, it is not the proof that made history but rather the intermediate step that he took to solve the problem: Euler's great insight lay in viewing Konigsberg's bridges as a *graph,* a collection of *nodes* connected by *links.* For this he used nodes to represent each of the four land areas separated by the river, distinguishing them with letters A, B, C, and D. Next he called the bridges the links and connected with lines those pieces of land that had a bridge between them. He thus obtained a graph whose nodes were pieces of land and links, were bridges.

Euler's proof that in Konigsberg there is no path crossing all seven bridges only once was based on a simple observation. Nodes with an odd number of links must be either the starting or the end point of the journey. A continuous path that goes through all bridges can have only one starting and one end point. Thus, such a path cannot exist on a graph that has more than two nodes with an odd number of links. As the Konigsberg graph had four such nodes, one could not find the desired path.

For our purpose the most important aspect of Euler's proof is that the existence of the path does not depend on our ingenuity to find it. Rather, it *is a property of the graph.* Given the layout of the Konigsberg bridges, no matter how smart we are, we will never succeed at finding the desired path. The people of Konigsberg finally agreed with Euler, gave up their fruitless search, and in 1875 built a new bridge between B and C, increasing the number of links of these two nodes to four. Now only two nodes (A and D) with an odd number of links remained. It was then rather straightforward to find the desired path. Perhaps the creation of this path was the hidden rationale behind building the bridge?

In retrospect, Euler's unintended message is very simple: Graphs or networks have properties, hidden in their construction, that limit or enhance our ability to do things with them. For more than two centuries the layout of Konigsberg's graph limited its citizens' ability to solve their coffeehouse problem. But a change in the layout, the addition of only one extra link, suddenly removed this constraint.

In many ways Euler's result symbolizes an important message of this book: The construction and structure of graphs or networks is the key to understanding the complex world around us. Small changes in the topology, affecting only a few of the nodes or links, can open up hidden doors, allowing new possibilities to emerge.

Graph theory boomed after Euler with contributions made by mathematical giants such as Cauchy, Hamilton, Cayley, Kirchhoff, and Polya. They uncovered just about everything that is known about large but ordered graphs, such as the lattice formed by atoms in a crystal or the hexagonal lattice made by bee? in a beehive. Until the mid-twentieth century the goal of graph theory was simple: It aimed to discover and catalogue the properties of the various graphs. Famous problems included finding a way to escape from a maze or labyrinth, first solved in 1873- or finding a sequence of moves with a knight on a chess board such chat each square is visited only once and the knight returns to its starting point. Some of the more difficult problems have gone unsolved for centuries.

Two centuries passed after Euler's inspiring work before mathematicians moved from studying the properties of various graphs to asking the quintessential question of how graphs, or, more commonly, net works, came about. Indeed, how do real networks form? What are the laws governing their appearance and structure. These questions, and the first answer, did not come until the 1950s, when two Hungarian mathematicians made a revolution in graph theory.

2.

One afternoon in late 1920s Budapest, a seventeen-year-old youth cantered with a weird gait through the streets and stopped in front of an elegant shoe shop that sold custom-made shoes. With his strangely shaped feet, on which normal shoes would never fit, he could indeed use a cobbler. But new shoes were not the occasion of this visit. After knocking on the store's door — an act that would have seemed just as odd back then as today — he entered, ignoring the saleswoman at the counter, and went up to a fourteen-year-old boy in the back of the shop.

"Give me a four digit number"' he said.

"2.532," came the wide-eyed boy's reply as he stared at the strange creature. The older boy did not let him stare too long; however.

"The square of it is 6,441,024," he continued. "'Sorry, I am getting old and I cannot tell you the cube. How many proofs of the Pythagorean theorem do you know?"

"One," replied the youngster.

"I know thirty-seven," and without taking a breath he continued. "Did you know that the points of a straight line do not form a countable set;" Alter showing the sharp boy Cantor's proof as evidence, his business at the cobbler's store finished, he said, "1 must run, " and so he did, turning on his heel and galloping out of the store.

Paul Erdös galloped on to become the presiding genius and most famous misfit of the twentieth century. He wrote more than 1,500 mathematics papers before his death in 1996. This output, unparalleled since Euler, contained eight articles published with another Hungarian mathematician, Alfred Rényi. These eight papers addressed for the first time in history the most fundamental question pertaining to our understanding of our interconnected universe: How do networks form' Their solution laid the foundation of the theory of random networks. This elegant theory so profoundly determined our thinking about networks that we are still struggling to break away from its hold.

3.

Organize a party for a hundred guests who have been selected and invited because they do not know a single other person on. the guest list-Offer this group of strangers wine and cheese, and they will immediately start to chat, as human beings' inborn desire to meet and know each other inevitably brings them together. Soon you will see thirty to forty groups of two or three. Now mention to one guest that the red wine in the unlabeled dark green bottles is a rare twenty-year-old vintage port, far better than that with the red label. But ask that guest to share this information only with bis or her new acquaintances. You know that your expensive port is fairly safe, because your friend has only had time to meet two or three people in the room. However, guests will inevitably become bored talking to the same person for too long and move on to join other groups. An outside observer would not notice anything special. Yet there are invisible social links between people who met earlier but now belong to different groups. As a consequence, subtle paths start connecting people who are still strangers to each other. For example, though John has not met Mary yet, they have both met Mike, and so there is a path from John to Mary through Mike. If John knew about the wine, chances are that now Mary knows too, since she could hear it from Mike, who was told by John. As time goes on, the guests will be increasingly interwoven by such intangible links, creating a fine web of acquaintances that include? a sizable portion of the guests. The expensive wine is increasingly endangered as its identity is passed from a tiny group of insiders to more and more chatting groups {Figure 2.2).

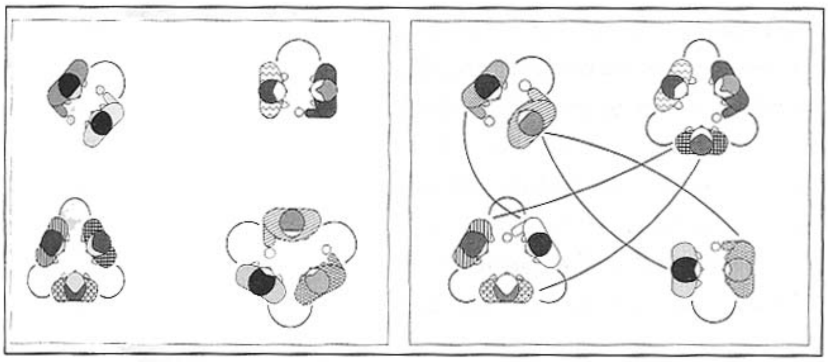

Figure 2.2 The Party, *At a party with ten guests, none of whom initially know one another, social ties form as the guests start chatting in small groups. At first, the groups are isolated from each other (left panel). Indeed, though there are social links (shown as continuous lines) between those in the same group, everyone outside of that group is still a stranger. As time goes on (right panel), three guests move to different groups and a giant cluster emerges. Although not everyone knows everyone else, there is now a single social network that includes all the guests. By following the social links, one can now find a path between any two guests.*

Assuming that each person passes on the information to all of her or his new acquaintances, will the reputation of the fine port reach all of the guests before the end of the party? To be sure, if all were to get to know each other, everybody would be pouring the superior wine from the unlabeled bottle. But even if each encounter took only ten minutes, meeting all ninety-nine others would take about sixteen hours. Parties rarely last that long. Thus, you might feel that you could reveal the identity of the wine to your friend and reasonably hope that some would be left at the end of the party.

Paul Erdös and Allied Rényi begged to differ. "A mathematician is a machine that turns coffee into problems" Erdo's used to say, quoting Rényi. A particularly lucky cup of coffee turned into a much quoted theorem: If each person gets to know at least *one* other guest, then soon everybody will be drinking the reserve port. According to Erdös and Rényi. it would take only thirty minutes to turn a single invisible social web that includes all guests in the room. Minutes after you hear the recommendation for the wine, you may rind yourself tipping an empty bottle into your expectant glass.

4.

The guests we met at the cocktail party arc part of a problem in graph theory, the branch of mathematics pioneered by Euler. The guests arc the nodes, and every encounter creates a social link between them. Thus a web of acquaintances — a graph — emerges, a bunch of nodes connected by links. Computers linked by phone lines, molecules in our body linked by biochemical reactions companies and consumers linked by trade, nerve cells connected by axons, islands connected by bridges are all examples of graphs. Whatever the identity and the nature of the nodes and links for a mathematician they form the same animal: a *graph* or a *network.*

Despite its elegance, simplifying all webs into graphs poses some formidable challenges. While society, the Internet, a cell, or the brain can all be represented by graphs, each is clearly very different from the others. It is hard to imagine much commonality between human society, where we make friends and acquaintances through a combination of random encounters and conscious decisions, and the cell, where the unforgiving laws of chemistry and physics govern all reactions between molecules. There must be a clear difference in the rules that govern the placement of links in the various networks we encounter in nature. Finding a model to describe all of these different systems seems, on its face, an insurmountable challenge.

Yet the ultimate goal of all scientists is to find the simplest possible explanation for very complex phenomena. Erdös and Rényi took on this challenge by proposing an elegant mathematical answer to describe all part of a single cluster such that, starting from any node, we can get to any other by navigating along the links between the nodes. This is the moment when your expensive wine is in danger, since a rumor can reach everyone who belongs to the giant cluster. Mathematicians call this phenomenon the emergence of a giant component- one that includes a large fraction of all nodes. Physicists call it percolation and will tell you that we just witnessed a phase transition, similar to the moment in which water freezes. Sociologists would tell you that your subjects had just termed a community. Though different disciplines may have different terminology, they all agree that when we randomly pick and connect pairs of nodes together in a network, something special happens: The network, after placing a critical number of links, drastically changes. Before, we have a bunch of tiny isolated clusters of nodes. disparate groups of people that communicate only within the clusters. *After,* we have a giant cluster, joined by almost everybody.

6.

Each of us is part of a large cluster, the worldwide social net-from which no one is left out. We do not know everybody on this globe, but it is guaranteed that there is a path between any two of us in this web of people. Likewise, there is a path between any two neurons in our brain, between any two companies in the world, between any two chemicals in our body. Nothing is excluded from this highly interconnected web of life. Paul Erdös and Alfred Kerryi told us why: It requires *only one link per node* to stay connected. One acquaintance per person, one link to at least one other neuron for each neuron in the brain, the ability to participate in at least one reaction for each chemical in our body, trade with at least one other company in the business world. One is the threshold. If nodes have less than one connection on average, then our network breaks into tiny noncommunicating clusters. If there is more than one connection per node, that danger becomes remote.

Nature repeatedly and extravagantly exceeds the one-link minimum. Sociologists estimate that we know between 200 and 5,000 people by name. An average neuron is connected to dozens of others, some to thousands. Each company is inevitably linked to hundreds of suppliers and customers; some of the biggest have links to millions. In our body, most molecules take part in tar more than a single reaction — some, like water, in hundreds. Thus, real networks not only are connected but are well beyond the threshold of one. Random network theory tells us that as the average number of links per node increases beyond the critical one, the number of nodes left out of the giant cluster decreases exponentially. That is, the more links we add, the harder it: is to find a node that remains isolated. Nature does not take risks by staying close to the threshold. It well surpasses it. Consequently, the networks around us are not just webs. They are very dense networks from which nothing can escape and within which every node is navigable. This is why there are no islands of people completely isolated from society at large and why all molecules in our body are integrated into a single complex cellular map. This is why the Apostle Paul's message reached people he never met and why MafiaBoy made headlines: Along the links their actions easily affected millions.

7.

Erdös and Rényi's discovery of this very special moment when a giant cluster emerges through a phase or percolation transition was a huge event in graph theory, bur not because it made the unbelievable prediction that only one acquaintance is required to form a society.

Rather, it was largely because, before Erdös and Reniyi, graph theory had not dealt with cocktail parties, social networks, or random graphs. It focused almost exclusively on regular graphs, which contain no ambiguity about their structure. But when it comes to such complex systems as the Internet or the cell, regular graphs ate the exception rather than the norm. Erdös and Rényi acknowledged for the first time that real graphs, from social networks to phone lines, are not nice and regular. They are hopelessly complicated. Humbled by their complexity, the two assumed that these networks are random.

In retrospect, it is not surprising that this unlikely pair of mathematicians were the ones to turn around a respectable field of mathematics by injecting randomness into it. Chance and randomness were very much a part of their lives. Though Rényi was seven years younger than Erdös, they knew each other thanks to the friendship between their parents back in Budapest. By the time they started working together, after meeting up in Amsterdam in 1948, both had lived through rather tumultuous time;. Subject to the *Numerus Clausus* laws that limited the number of Jews admitted to university, Rényi had worked in a shipyard after high school. After winning a math and Greek competition, he was allowed to enter the university in 1939. Soon after finishing his mathematical studies he was called to forced labor, from which he somehow escaped.

Erdös and his colleagues, who were familiar with Rényi's resistance activities during the war, deeply admired and respected him. Rényi had boldly disguised himself in the uniform of the Hungarian fascists, Kyilas, to help his friends escape the concentration camps. According to one story, Rényi entered the Budapest ghetto dressed as a Kyilas soldier and managed to escort his parents out. He also lived for years in Nazi-controlled Budapest using false documents. Only those aware of the realities of the Nazi terror could truly appreciate the courage needed to perform these acts. Not surprisingly, Rényi's ability to focus on mathematics was highly constrained until the end of the war, when in 1946 he traveled to Leningrad to continue his studies. There his creativity exploded. He not only learned and absorbed number theory in record time, despite his limited Russian language skills, but also proved some fundamental theorems on one of the notoriously difficult problems of number theory, the Goldbach conjecture. Thus, when he met Erdös two years later in Amsterdam, he was no longer the aspiring young mathematician and family friend but a well-known scientist with an international reputation.

Erdös by then had already developed his trademark traveling-mathematician lifestyle. He would show up at his colleagues' doorsteps and proclaim, "My brain is open," an invitation to join in his tireless pursuit of mathematical truth. His only permanent job offer came from the University of Notre Dame, in South Bend, Indiana. Arnold Ross, at that time the chairman of the math department, offered Erdös a visiting professorship on very generous terms: He could come and go as he pleased, since he had an assistant who would pick up the lectures where he left them oft.

A Catholic liberal arts college, Notre Dame was not the prominent university it would become decades later. Nevertheless it offered Erdös a quiet and comfortable work environment and the opportunity for frequent discussions with his priest colleagues, which Erdös, with his unique perspective on the universe and deity, particularly enjoyed. Once asked about his time there, he remarked tongue-in-cheek, "There are too many plus signs" a reference to the numerous crucifixes about campus. When Notre Dame eventually offered to turn Erdös's status into a permanent one, on the same comfortable terms, Erdös politely refused. Perhaps losing the randomness and unpredictability that had characterized his life was too much for him to fathom.

8.

The Amsterdam meeting between Erdös and Rényi was the start of a very close friendship and collaboration that resulted in over thirty joint publications before Rényi's early death at the age of forty-nine in 1970. Among these publications were the eight legendary papers on graph theory. The first, published more than a decade after the Amsterdam meeting, addressed for the first time the important questions of how graphs form. Their use of randomness to tackle graph theory problems is most evident when we look at how many links nodes have in a graph or network. Regular graphs are unique in that each node has *exactly* the same number of links. Indeed, in a two-dimensional mesh of perpendicular lines forming a simple square lattice each node has exactly four links, or in a hexagonal lattice of a beehive each node is connected to exactly three others.

Such regularity is clearly absent from random graphs. The premise of the random network model is deeply egalitarian: We place the links completely randomly; thus all nodes have the same chance of getting one — just as in Las Vegas, where supposedly we all have the same chance of hitting the jackpot. At the end of the day, however, only a few of our fellow gamblers walk away richer. Similarly, if we place the links randomly in a graph, some nodes will get more links than others. Some might even have bad luck and get nothing for a while. The random world of Erdös and Rényi can he simultaneously unfair and generous: It can make some poor and others rich. Yet a far-reaching prediction of Erdös and Rényi's theory tells us that this only appears to be so. If the network is large, despite the links' completely random placement, almost *all nodes will have approximately the same number of links.*

One way to see this is to interview all guests as they leave the cocktail party, asking them how many acquaintances they made. When everybody leaves, we can draw a histogram by plotting how many of the guests have one, two, or exactly *k* new acquaintances. For the random network model of Erdös and Rényi the shape of the histogram was derived and proved exactly in 1982 by one of Erdös's students, Bela Bollobas, professor of mathematics at the University of

Memphis in the United States and Trinity College in the United Kingdom. The result shows that the histogram follows a Poisson distribution, which has some unique properties that will follow us throughout this book, A Poisson distribution has a prominent peak, indicating that the majority of nodes have the same number of links as the average node does. On the two sides of the peak the distribution rapidly diminishes, making significant deviations from the average extremely rare.

Translated back to a society of 6 billion people, a Poisson distribution tells us that most of us have roughly the same number of friends and acquaintances. It predicts that it is exponentially rare to find someone who deviates from the average by having considerably more or fewer links than the average person. Therefore, random graph theory predicts that if we assign social links randomly, we end up with an extremely democratic society, where all of us are average and very few deviate from the norm to be extremely social or utterly asocial types. We obtain a network with a very uniform fabric in which the mean is the norm.

Erdös and Rényi's random universe is dominated by averages. It predicts that most people have roughly the same number of acquaintances; most neurons connect roughly to the same number of other neurons; most companies trade with roughly the same number of other companies; most Websites are visited by roughly the same number of visitors. As nature blindly throws the links around, in the long run no node is favored or singled out.

9.

The random network theory of Erdös and Rényi has dominated scientific thinking about networks since its introduction in 1959. It created several paradigms that are consciously or unconsciously imprinted on the minds of everyone who deals with networks. It equated complexity with randomness. If a network was too complex to be captured in simple terms, it urged us to describe it as random. Sure enough, society, the cell, communication networks, and the economy are all complex enough to fit the bill.

You may be thinking that there is something fishy about this random universe, in which all nodes are equal. Would I be able to write this book if the molecules in my body decided to react to each other randomly? Would there be nations, states, schools and churches or any other manifestations of social order if people interacted with each other completely randomly: Would we have an economy if companies selected their consumers randomly, replacing their salespeople with millions of dice? Most of us *feel* that we do not live in such a random world — that there has to be some order behind these complex systems.

Why, then, would two such unparalleled intellects as Erdös and Rényi choose to model the emergence of networks as a completely random process? The answer is simple: They never planned to provide a universal theory of network formation. They were far more intrigued by the mathematical beauty of random networks than by the model's ability to faithfully capture the webs nature created around them. To be sure, in their seminal 1959 paper they did mention that "the evolution of graphs may be considered as a rather simplified model of the evolution of certain communication nets (railway, road or electric network systems, etc.)." But, despite this brief journey into the real world, their work in this area was motivated by a deep curiosity about the mathematical depths of the problem rather than by its applications.

Erdös would be the first to agree with us that real network; must have organizing principles that distinguish them from the random network model they introduced in 1959. But for him this would be beside the point. By using the hypothesis of randomness he opened a window to a new world, whose mathematical beauty and consistency was the main driving force behind the subsequent work in graph theory.

Until recently we had no alternative for describing our interlinked universe. Thus random networks came to dominate our ideas on network modeling. Complex real networks were viewed as fundamentally random.

Erdös holds the record for suggesting good problem; and making sure that somebody else solved them. Though he never owned more than a few clothes that fit into a small leather suitcase that he always traveled with, he often offered monetary rewards for solutions or proofs to problems that he found interesting — $5 for a problem he considered simple, $500 for a truly difficult one. And he would happily pay if the proof was delivered. Never mind that often a $1 problem turned out to be more difficult than a $500 one. The lucky mathematicians who earned one of his rewards never cashed his checks anyway. Most of them framed them. The reward was a unique recognition by the presiding genius of the century; no cash amount could match its spiritual value.

Let us follow Erdös's example and ask a question he left untouched. What do *real* networks look like? Posing a problem in such a sloppy way would never have satisfied him. It is too broad. It may not even have a unique answer. And most likely we can never offer a rigorous proof. Thus it could not possibly be from the Transfimite Book, the ultimate depository in Erdös's world of all good mathematical proofs and theorems. But though the question might not have won his approval, in the coming chapters, we will see that it makes a huge difference outside the world of mathematics.